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Vol: 2 & Iss: 1

Research Article

Analysis and Control of Plasma Turbulence

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Citation: Sridhar LN. Analysis and Control of Plasma Turbulence. *Arch Wom Health 2026;* 2(2): 113-124.

Received: 14 April, 2026; **Accepted:** 21 April, 2026; **Published:** 23 April, 2026

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A B S T R A C T

Bifurcation analysis and Multiobjective Nonlinear Model Predictive Control is performed on the Sugama Horton and the Ball Dewar Sugama plasma turbulence models. The MATLAB program MATCONT was used to perform the bifurcation analysis. The MNLMPC calculations were performed using the optimization language PYOMO in conjunction with the state-of-the-art global optimization solvers IPOPT and BARON. The bifurcation analysis revealed the existence of Hopf bifurcation points and branch points. The MNLMC converged on the Utopian solution. Hopf bifurcation points, which cause unwanted limit cycles, are eliminated using an activation function based on the tanh function. The branch points (which cause multiple steady-state solutions from a singular point) are very beneficial because they enable the Multiobjective Nonlinear Model Predictive Control calculations to converge to the Utopia point (the best possible solution) in the model.

Keywords: Bifurcation, Optimization, Control, Acetic-anhydride, CSTR

1. Background

In this manuscript, a comprehensive, engineering-focused analysis of plasma turbulence is conducted by applying the Sugama-Horton and Ball-Dewar-Sugama reduced-order models to identify the relevant transport processes directly applicable to industrial energy systems. By integrating nonlinear dynamics, bifurcation dynamics, stability boundaries, and analysis, the manuscript identifies controllable routes for mitigating turbulence and optimizing transport. The manuscript applies advanced plasma physics to a process systems context.

There is a wide spectrum of plasma science and engineering applications in economies that are critical to national priorities, including magnetic confinement plasmas, plasma manufacturing, space and astrophysical plasmas, and environmental plasmas. There is an intense nonlinearity at many scales in plasma systems as a result of a wide array of interacting particles, fields, waves, and flows-a characteristic that defines plasmas as interacting systems. These non-linear effects include turbulent behavior, transport barrier formation, self-organization, and sudden changes in operational regimes. Yet, a critical issue in plasma system control is that small variations in parameters or disturbances can cause major qualitative changes in plasma system behavior, thereby destabilizing system operations and performance.

Natural ecosystems provide a powerful conceptual framework for investigating the complexity of these systems. Ecosystems are composed of diverse, interacting components that have evolved through nonlinear feedback, adapting to changing environments while maintaining functionality through distributed regulation rather than centralized control. Crucially, ecosystems balance multiple competing objectives-efficiency, robustness, adaptability, and sustainability-while remaining resilient to external perturbations. These features closely parallel the operational demands of plasma systems, motivating an ecosystem-inspired approach to plasma modeling, analysis, and control.

This project will outline a novel, ecosystem-inspired innovation framework for plasma science and engineering that integrates bifurcation analysis with multi-objective nonlinear model-predictive control. Plasma subsystems are treated as interacting agents within a dynamic ecosystem, including turbulence modes, zonal flows, energetic particle populations, and electromagnetic fields. Their nonlinear interactions lead to emergent collective behavior and critical transitions that govern overall system performance. Bifurcation analysis is conducted to systematically identify the stability boundaries, tipping points, and regime transitions as control parameters and operating conditions vary. Mapping these bifurcation structures will provide mechanistic insight into confinement transitions, turbulence suppression, and loss-of-stability events that are central to plasma operation.

Leveraging advances in dynamical understanding, the proposed research aims to integrate multi-objective approaches into NMPC strategies that explicitly account for nonlinear plasma physics and ecosystem-inspired control. Contrary to existing linear or other multi-objective control approaches, in which a few possibly conflicting objectives may be considered, in the proposed new framework, the NMPC strategy tends to simultaneously optimize several objectives, such as maximizing plasma confinement/throughput, reducing energy consumption, and ensuring robustness with respect to disturbances, while taking into account the nonlinear plasma physics. Moreover, by incorporating bifurcation analysis capabilities into the predictor part of the control scheme, the NMPC controller is likely to be able to detect and act in real time to prevent unwanted plasma bifurcations or to deliberately induce wanted ones.

Such a closed-loop, anticipatory control concept is analogous to regulation in ecosystems, where individual, local processes collectively contribute to global stability. The combination of bifurcation analysis with multi-objective NMPC provides a means of pro-actively controlling plasma systems, converting hitherto uncontrollable changes between regimes into beneficial control options.

Plasmas confined by magnets display nonlinear dynamics that result from the interplay of turbulence, flows, and gradients. Such nonlinear dynamics play a critical role in limiting the predictability and controllability of the plasma device's behavior. In particular, the transition from low-confinement mode (L-mode) to high-confinement mode (H-mode), which is critical in tokamaks and stellarators, remains an open problem in plasma physics and in the applications of controlled thermonuclear devices.

Reduced plasma theory models have been critical to understanding the underlying principles of these transitions. Among the more dominant models have been the Sugama-Horton turbulence shear flow model[1]. and extensions of these have been carried out in subsequent studies by Ball and Dewar[2] These describe the nonlinear coupling between turbulence intensity, shear flow, and pressure gradients. The low-dimensional character of these approaches is evidenced by their ability to accurately capture various experimentally observed phenomena, including bifurcation-driven transitions, hysteresis, and oscillations.

They have a rather natural home in the theory of nonlinear dynamical systems with several control parameters, with confinement transitions taking the forms of pitchfork and transcritical bifurcations, together with Hopf bifurcations in the cases of magnetically confined plasmas. Ball and Dewar[2] have shown that singularity theory offers a unified view of transitions, revealing how changes in plasma parameters qualitatively affect its behavior. Aspects of existence, uniqueness, and global stability have been studied analytically, confirming the mathematical well-posedness of the plasma-dynamics models.

A primary challenge in the science of magnetized plasmas, in the context of magnetic confinement fusion as well as in space plasma physics, is understanding and predicting nonlinear dynamics, particularly in regulating transport, generating zonal flows, and self-organization. Certainly, reduced theoretical models that preserve fundamental physical structures, such as conservation laws, invariants, and variational principles, are highly valuable tools for the analysis and control of plasmas. Two widely popular mathematical models are the so-called Sugama–Horton drift-wave/zonal-flow models, as well as the Ball–Dewar variational/relaxation models, which specifically deal with plasma dynamics, particularly in terms of nonlinear turbulence-flow interaction and energy-constrained self-organization, respectively. The models can be classified as

- Drift-Wave/Zonal-Flow Reduced Models
- Ancestor Reduced Models: Hasegawa–Mima and Hasegawa–Wakatani
- Variational and Hamiltonian Plasma Theory
- Variational/Relaxation Models: Ball–Dewar Framework
- Multi-Region Relaxed MHD (MRxMHD): Bridging Dynamics and Energy Constraints

A brief description of these models will now be presented.

2. Drift-Wave/Zonal-Flow Reduced Models

One of the fundamental mechanisms at work in confined plasmas involves the interaction between drift wave turbulence and zonal flows, and its rather strong influence on the actual level of transport as well as the creation of transport barriers. The Sugama-Horton models constitute an exemplary class of reduced descriptions of plasmas, and, based on these, one can model the interaction via coupled nonlinear field equations. These Sugama-Horton models also display corresponding predator-prey type dynamics with regard to turbulence and zonal shear flows, and one can apply these to corresponding bifurcation scenarios and transitions with respect to transport levels. Sugama, Horton, and Dewar[3] extended these earlier Sugama-Horton descriptions in order to more specifically analyze the underlying dynamics with respect to L-H-I transit.

Sugama & Horton[1] presented a nonlinear reduced model that shows self-regulation of the drift-wave turbulence by zonal flows and elucidated mechanisms of transport bifurcations in parameter space. These models belong to the family of Hamiltonian-reducible theories that keep the essential conservation laws and symmetries and are therefore natural candidates for a rigorous analysis and control design. A general theoretical background of such turbulence reduced models is surveyed in Diamond, et al.[4] where also mechanisms of generation and nonlinear saturation of zonal flows are discussed anchoring many investigations of reduced models within a single conceptual framework.

3. Ancestor Reduced Models: Hasegawa-Mima and Hasegawa-Wakatani Model

A reduced equation for drift-wave turbulence was formulated by Hasegawa & Mima[5] and exhibits a Hamiltonian structure with nonlinear wave coupling. Additional physics, such as the effects of resistive coupling, were introduced by Hasegawa & Wakatani[6] and provide the basis for these edge-turbulence phenomena, serving as the basis for many reduced-turbulence models, often used as canonical case studies for analytic bifurcation theory and pattern dynamics.

4. Variational and Hamiltonian Plasma Theory

There has been a unifying mathematical framework for reduced plasma models from the perspective of noncanonical Hamiltonian theory and energy-Casimir theory. Morrison & Greene proved the noncanonical formalism for ideal magnetohydrodynamics as a basis for the development of structure-preserving reduced models[7]. Morrison reviewed the various Hamiltonian structures that have been found for reduced plasma models and the other reduced models of various fluid systems, emphasizing the capability of reduced models to preserve the invariant properties needed for studies of plasma stability and bifurcation theory[8]. Energy-Casimir stability theory, as proved by Holm, et al.[9], has been used to study the equilibria of reduced models. These are the tools on which the building of the Sugama-Horton-type turbulence models and the variational self-organization models are based.

5. Variational/Relaxation Models: Ball-Dewar Framework

The variational or relaxation models complement the dynamical turbulence models in that they look for equilibria subject to invariants and energetic constraints. These formulations make explicit how plasmas self-organize into structured states under constraints imposed by conservation laws. Dewar[10] was the first to point out how instabilities and relaxation processes interact in driving plasma to constrained equilibrium states. Ball & Dewar[2] codified the influence of global invariants on the state of self-organized plasma, developing variational principles that explicitly couple macroscopic structure to the underlying invariants. Subsequent generalizations by Dewar and others[11] have led to variational formulations that incorporate transport effects and a complex constraint structure. These variational models are critical to understanding plasma equilibrium selection in the presence of competing dynamical processes and also provide fertile ground for stability and control analyses.

6. Multi-Region Relaxed MHD (MRxMHD): Bridging Dynamics and Energy Constraints

Researchers have recently extended the variational principles of relaxation to the multi-region equilibrium concept, in which regions of the plasma, defined by topological features, independently relax subject to constraints on conserved quantities. This leads to several complex equilibrium structures that share many characteristics with self-organizing structures observed in experimental devices. Hudson, et al.[12]. developed algorithms to solve the multi-region relaxed MHD equilibrium problem, enabling the inclusion of a wide variety of both conservation and geometrical constraints. Dewar, et al.[13]. further extended the principles of the variational problem to articulate the process of self-organization within the multi-domain equilibrium context. Within the field of MHD, the MRxMHD has provided a bridge between the more recently developed dynamic models of turbulence-flow interactions and the more traditional constrained models of the structure of the system.

7. Low-Dimensional Transport Models and Predator-Prey Analogies

To facilitate analytical tractability and controllability, several researchers have investigated reduced frameworks that abstract relevant interactions within predator–prey-type systems. As a result, this is highly analogous to the relevant reduced system and provides a good intuitive understanding of bifurcation results. Diamond and Kim[14] considered the generation of mean poloidal flow through a study of drift wave turbulence, hence establishing a basis for reduced interactions. Itoh, et al.[15] have given a comprehensive review on reduced transport models, including structural bifurcation results as well as hysteresis, hence capturing a number of relevant phenomenological results of great interest to edge turbulence as well as the formation of transport barriers.

8. Reduced Gyrofluid and Gyrokinetic Representations

The reduced gyrofluid models are intermediate, structure-preserving models that maintain finite-Larmor-radius effects and kinetic closure while reducing complexity.

Beer & Hammett[16] have identified the toroidal gyrofluid equations developed for the simulation of plasma turbulence, which yielded a set of closed moment equations with energy properties. Sugama & Horton (1998)[17] developed gyrofluid approaches with specific applications to toroidal plasmas, in which kinetic physics is linked to reduced-turbulence dynamics. This relates to the continuum from fully kinetic approaches to reduced approaches, such as the Sugama-Horton approaches.

Taken together, the literature reveals a rich ecosystem of plasma models that preserve key physical structures-invariants, conservation laws, and variational principles-while enabling tractable analysis of nonlinear dynamics. The Sugama-Horton frameworks occupy a central position among reduced-turbulence–flow-interaction models, and the Ball–Dewar principles provide a complementary variational perspective.

Ball, et al.[18] studied the metamorphosis of plasma turbulence-shear-flow dynamics through a transcritical bifurcation reducing the plasma dynamic model to an ecological model.

The plasma turbulence-shear flow analogy in ecology has been well studied and is used as an exact mathematical analogy for the behavior of turbulence-regulating systems, during excursions, and as a detailed mathematical analogy for the phenomenon of "ecological collapse." It establishes that the competition of turbulent energy (drift waves) with enhanced sheared mean flows (zonal flows) is mathematically similar to a predator-prey system with shear flows as the predator and the turbulent fluctuations as its prey.

Although bifurcation analysis has significantly advanced the scientific understanding of plasma confinement transitions, the exploitation of such bifurcation structures for control has, to a large extent, not been investigated. Indeed, plasma control is currently based on stabilizing a desired state without properly accounting for the plasma's overall nonlinear structure.

Bifurcation analysis and Multiobjective Nonlinear Model Predictive Control (MNLMPC) are typically performed separately for nonlinear problems. The aim of this paper is to provide a mathematical framework involving the integration of bifurcation analysis and multiobjective nonlinear model predictive control calculations for the Sugama Horton[1] and Ball Dewar Sugama Ball, et al.[18] plasma models. The paper is organized as follows. First, the model equations for both models are presented, followed by a discussion of the numerical techniques involving bifurcation analysis and multiobjective nonlinear model predictive control (MNLMPC). The results and discussion are then presented, followed by the conclusions.

9. Description of Model Equations

9.1. Original sugama horton model

The Sugama-Horton model has its roots in basic fluid models of magnetized plasmas, beginning with the conservation of ion and electron mass and momentum. One begins with a two-fluid model, assuming strong magnetization and low-frequency electrostatic modes characteristic of drift wave turbulence. By taking advantage of the scale separation between fast gyro-phase variations and slow cross-field evolution, the equations are reduced using drift ordering. The quasi-neutrality is imposed to relate density and potential fluctuations, while electron motion along magnetic field lines yield a non-adiabatic response that leads to instability. Ion polarization drift currents introduce vorticity evolution, relating flow shear and density gradients. Collisional terms and Larmor radius corrections are included to account for stabilizing effects. By systematic asymptotic reduction and normalization, the original set of equations is reduced to a controlled-fluid model of drift-wave turbulence, focusing on nonlinear interactions, shear stabilization, and energy transfer in magnetically confined plasmas.

In this model[1], pv is the mean pressure gradient (free energy reservoir), nv is the turbulence intensity (drift-wave energy), and fv represents the Zonal/shear flow energy (turbulence predator), q is the heating rate (external power input), γ is the drive coefficient(gradient turbulence coupling) , α is the shear suppression coefficient (turbulence flow coupling), β (nonlinear saturation that causes turbulence self-damping), μ_0, μ_1, μ_2 collisional damping, pressure-dependent damping, and turbulence drag.

The equations are

$$\mu(pv,nv) = \mu_0 + \mu_1(pv) + \mu_2(pv^2)$$
$$\frac{dP}{dt} = q - \gamma pv(nv)$$
$$\frac{dN}{dt} = \gamma pv(nv) - \alpha fv(nv) - \beta(nv)^2$$
$$\frac{dF}{dt} = \alpha fv(nv) - \mu(pv,nv)fv$$

The base parameter values are

$$q = 1.05, \gamma = 1.25, \alpha = 1.75, \beta = 1.25, \mu_0 = 0.55, \mu_1 = 0.5, \mu_2 = 0.5.$$

9.2. Ball dewar sugama model (BDS model)

The Ball-Dewar-Sugama model is obtained from the basic fluid description of a magnetized plasma, starting from the conservation equations of mass, momentum, and charge for ions and electrons. In the presence of a strong background magnetic field and low-frequency fluctuations, the fluid equations are reduced using drift ordering, distinguishing between the fast gyro-motion and the slower cross-field transport. Quasi-neutrality relations connect the density and electrostatic potential fluctuations, while parallel electron motion along magnetic field lines introduce a non-adiabatic response susceptible to drift-wave instability. Ion polarization contributions enter the vorticity evolution equation, relating flow shear to pressure and density gradients. The Ball and Dewar model improves the modeling of nonlinear coupling and energy transfer between drift waves and zonal flows, with a focus on the self-regulation of turbulence. Dissipation and collisional terms are added to provide realistic saturation dynamics. By systematic reduction and normalization, the final model retains the dominant physics of turbulence amplification, nonlinear saturation, and shear suppression in a confined plasma.

In this model[12], pv is the pressure gradient that drives the turbulence, nv represents the turbulence intensity that can generate flows, and fv s the shear flow energy, and $\sqrt{fv} = v$, the shear flow velocity, $\mu(pv,nv)$ is the v viscous damping of shear flow, q is the external power input, γ represents the turbulent flow rate constant, α is the energy transfer coefficient from turbulence to shear flow, β the nonlinear turbulence damping coefficient, φ is the external shear flow drive, while a, b are the viscosity constants and m, r are the exponents in viscosity. ε is the time scale separation factor.

The three equations are

$$\varepsilon\frac{dP}{dt} = q - \gamma pv(nv)$$
$$\frac{dN}{dt} = \gamma(pv)nv - \alpha fv(nv) - \beta(nv)^2$$
$$\frac{dF}{dt} = \alpha fv(nv) - \mu(pv,nv)fv + \varphi\sqrt{fv}$$
$$\mu(P,N) = bpv^m + apv^r(nv)$$

In the first equation, q adds energy to the pressure and $\gamma pv(nv)$ is the loss of pressure to turbulence. In the second equation, $\gamma(pv)nv$ increases turbulence, $\alpha fv(nv)$ suppresses turbulence and $\beta(nv)^2$ the nonlinear saturation. In the third equation, $\alpha(fv)nv$ represents the energy input from the turbulence, and $\varphi\sqrt{F}$ the external drive. The base values are

$$q = 2.5, \gamma = 1, \alpha = 2.4, \beta = 1, m = -3/2\ r = 1,\ a = 0.3, b = 18.58,\ \varphi = 0.01$$

9.3. Bifurcation analysis

Bifurcation analysis deals with multiple steady-states (caused by branch and limit points) and limit cycles, which are caused by Hopf bifurcation points. The MATLAB program MATCONT[19,20] is used to locate limit points, branch points, and Hopf bifurcation points. In ODE system

$$\frac{dx}{dt} = f(x, \alpha)$$

$x \in R^n$ Let the bifurcation parameter be α. Since the gradient is orthogonal to the tangent vector, The tangent plane is the n+1-dimensional vector w that satisfies

$$Aw = 0$$

Where A is

$$A = [\partial f / \partial x \quad | \partial f / \partial \alpha]$$

And $\partial f / \partial x$ is the Jacobian matrix. For both limit and branch points, the Jacobian matrix $J = [\partial f / \partial x]$ must be singular.

For a limit point, there is only one tangent at the point of singularity. At this singular point, there is a single non-zero vector, y, where Jy=0. This vector is of dimension n. Since there is only one tangent the vector

$y = (y_1, y_2, y_3, y_4, ...y_n)$ must align with $\hat{w} = (w_1, w_2, w_3, w_4, ...w_n)$. Since

$$J\hat{w} = Aw = 0$$

the n+1 th component of the tangent vector w_{n+1} = 0 at a limit point (LP).

For a branch point, there must exist two tangents at the singularity. Let the two tangents be z and w. This implies that

$$Az = 0$$

$$Aw = 0$$

Consider a vector v that is orthogonal to one of the tangents (say w). v can be expressed as a linear combination of z and w ($v = \alpha z + \beta w$). Since $Az = Aw = 0$; $Av = 0$ and since w and v are orthogonal, $w^T v = 0$. Hence $Bv = \begin{bmatrix} A \\ w^T \end{bmatrix} v = 0$ which implies that B is singular.

Hence, the matrix $B = \begin{bmatrix} A \\ w^T \end{bmatrix}$ is singular at a branch point.

When there is a Hopf bifurcation point the bialternate product,

$$\det(2 f_x(x, \alpha) @ Jn) = 0$$

where Jn is the n-square identity matrix. Hopf bifurcations cause limit cycles and should be eliminated because limit cycles make optimization and control tasks very difficult. More details can be found in Kuznetsov[21-23].

Hopf bifurcations cause limit cycles. Limit cycles cause equipment damage and make control tasks difficult. Additionally, they result in less beneficial products. The tanh activation function (where a control value v is modified to $(v \tanh v / \varepsilon)$ is used to eliminate spikes in profiles[24-27]. Sridhar[28] demonstrates with several examples how the activation factor involving the tanh function successfully eliminates the limit cycle causing Hopf bifurcation points by increasing the oscillation time period in the limit cycle.

10. Multiobjective Nonlinear Model Predictive Control (MNLMPC)

The rigorous multiobjective nonlinear model predictive control (MNLMPC) method developed by Flores Tlacuahuaz, et al.[29] was used. Consider a problem where the variables $\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i)$ have to be optimized together for a dynamic problem

$$\frac{dx}{dt} = F(x, u)$$

t_f being the final time value and u the control parameter. The individual objective optimal control problem is solved by optimizing each of the variables $\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i)$ The optimization of $\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i)$ will lead to the values q_j^*. Then the multiobjective optimal control (MOOC)problem

$$\min(\sum_{j=1}^{n}(\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i) - q_j^*))^2$$

$$subject\ to\ \frac{dx}{dt} = F(x, u);$$

is solved. This will provide the value of u at each time step. The first obtained control value of u is implemented and this procedure is repeated until the implemented and the first obtained control values are the same or where ($\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i) - q_j^* = 0$ for all j. Utopia point) is obtained.

The optimization program PYOMO[30] is used. Here, the differential equations are converted to a Nonlinear Program (NLP) using the orthogonal collocation method. The NLP is solved using IPOPT[31] and confirmed as a global solution with BARON[32].

The steps of the algorithm are as follows

- Optimize $\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i)$ and obtain q_j^*.
- Minimize $(\sum_{j=1}^{n}(\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i) - q_j^*))^2$ and get the control values at various times.
- Implement the first obtained control values
- Repeat steps 1 to 3 until there is an insignificant difference between the implemented and the first obtained value of

the control variables or if the Utopia point is achieved. The

Utopia point is when $\sum_{t_{i=0}}^{t_i=t_f} q_j(t_i) - q_j^* = 0$ for all j.

Sridhar[33] demonstrated that when the bifurcation analysis revealed the presence of limit and branch points the MNLMPC calculations to converge to the Utopia solution. This was done by imposing the singularity condition, caused by the presence of the limit or branch points on the co-state equation[34].

11. Results and Discussion

11.1. Theorem

If one of the functions in a dynamic system is separable into two distinct functions, a branch point singularity will occur in the system.

11.2. Proof

Consider a system of equations

$$\frac{dx}{dt} = f(x, \alpha)$$

$x \in R^n$. Defining the matrix A as

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} \cdots\cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial \alpha} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} \cdots\cdots & \frac{\partial f_2}{\partial x_n} & \frac{\partial f_2}{\partial \alpha} \\ \cdots\cdots\cdots\cdots & & & & & \\ \cdots\cdots\cdots\cdots & & & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \frac{\partial f_n}{\partial x_3} & \frac{\partial f_n}{\partial x_4} \cdots\cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial \alpha} \end{bmatrix}$$

α is the bifurcation parameter. The matrix A can be written in a compact form as

$$A = [\frac{\partial f_p}{\partial x_q} . | \quad \frac{\partial f_p}{\partial \alpha}]$$

The tangent at any point x; ($z = [z_1, z_2, z_3, z_4, \ldots z_{n+1}]$) must satisfy

$$Az = 0$$

The matrix $\{\frac{\partial f_p}{\partial x_q}\}$ must be singular at both limit and branch points. The n+1 th component of the tangent vector z_{n+1} = 0 at a limit point (LP) and for a branch point (BP) the matrix $B = \begin{bmatrix} A \\ z^T \end{bmatrix}$ must be singular. Any tangent at a point y that is defined by $z = [z_1, z_2, z_3, z_4, \ldots z_{n+1}]$) must satisfy

$$Az = 0$$

For a branch point, there must exist two tangents at the singularity. Let the two tangents be z and w. This implies that

$$Az = 0$$
$$Aw = 0$$

Consider a vector v that is orthogonal to one of the tangents (say z). v can be expressed as a linear combination of z and w ($v = \alpha z + \beta w$). Since $Az = Aw = 0$; $Av = 0$ and since z and v are orthogonal,

$z^T v = 0$. Hence $Bv = \begin{bmatrix} A \\ z^T \end{bmatrix} v = 0$ which implies that B is singular where $B = \begin{bmatrix} A \\ z^T \end{bmatrix}$

Let any of the functions f_i are separable into 2 functions ϕ_1, ϕ_2 as

$$f_i = \phi_1 \phi_2$$

At steady-state $f_i(x, \alpha) = 0$ and this will imply that either $\phi_1 = 0$ or $\phi_2 = 0$ or both ϕ_1 and ϕ_2 must be 0. This implies that two branches $\phi_1 = 0$ and $\phi_2 = 0$ will meet at a point where both ϕ_1 and ϕ_2 are 0.

At this point, the matrix B will be singular as a row in this matrix would be

$$[\frac{\partial f_i}{\partial x_k} | \frac{\partial f_i}{\partial \alpha}]$$

However,

$$[\frac{\partial f_i}{\partial x_k} = \phi_1(=0)\frac{\partial \phi_2}{\partial x_k} + \phi_2(=0)\frac{\partial \phi_1}{\partial x_k} = 0 (\forall k = 1.,, n)$$
$$\frac{\partial f_i}{\partial \alpha} = \phi_1(=0)\frac{\partial \phi_2}{\partial \alpha} + \phi_2(=0)\frac{\partial \phi_1}{\partial \alpha}] = 0$$

This implies that every element in the row $[\frac{\partial f_i}{\partial x_k} | \frac{\partial f_i}{\partial \alpha}]$ would be 0, and hence the matrix B would be singular. The singularity in B implies that there exists a branch point.

12. Bifurcation Analysis (Sugama Horton Model)

When α is the bifurcation parameter (the other parameter values are the base values) , a branch point was located at (P,N,F, α) **(Figure 1a)** values of (0.632456, 0.632456. 0, 1.685854) the two distinct equations can be obtained from the third ODE.

$$\frac{dF}{dt} = \alpha fv(nv) - \mu(pv, nv) fv$$

The two distinct equations are

$$fv = 0$$
$$\alpha(nv) - \mu = 0$$

$\mu(pv,nv) = \mu_0 + \mu_1(pv) + \mu_2(pv)^2$;
$\mu_0 = 0.55, \mu_1 = 0.5, \mu_2 = 0.5$.. P= 0.632456; N= 0.632456;. This implies that μ(pv,nv)=1.0662 $\alpha = 1.685854; \alpha N = 1.0662$ Both equations are satisfied, thereby validating the theorem.

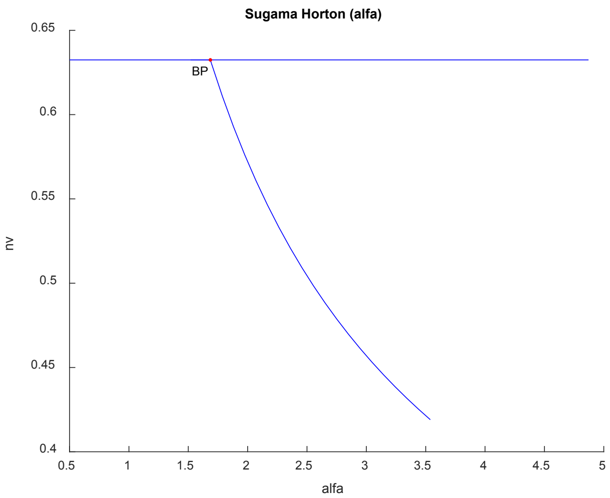

Figure 1a: Sugama Horton model Bifurcation Analysis α is the bifurcation parameter.

With β the bifurcation parameter and $\gamma = 0.85$ and all other parameter values being the same as the base values, a branch point, a Hopf bifurcation point, and a branch point were located at '(pv,nv,fv, β) values (0.805781, 0.730019, 0.345504, 0.337825) and (0.805781, 0.730019 0, 1.148163) **(Figure 1b)**. The limit cycle produced by this Hopf bifurcation point is seen in **(Figure 1c)**. When the bifurcation parameter was modified to $\beta \tanh(\beta)/0.0008$ a branch point is located at (pv,nv, fv, β) Values of (0.805781, 0.730019, 0, 0.027400), but the Hopf bifurcation point disappears, validating the analysis of Sridhar[28] **(Figure 1d)**.

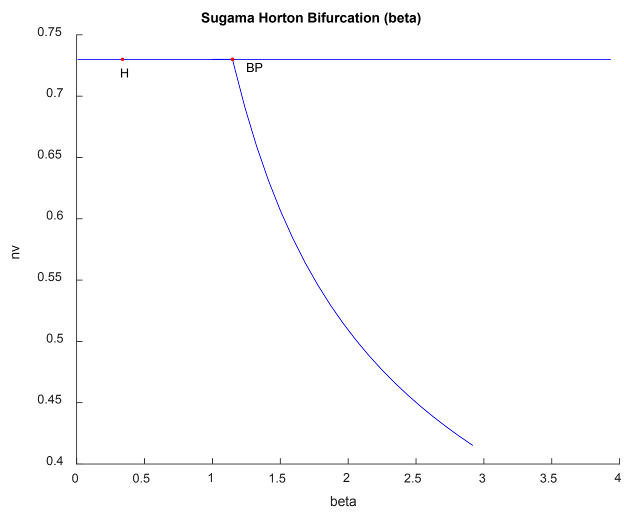

Figure 1b: Sugama Horton model Bifurcation Analysis β is bifurcation parameter.

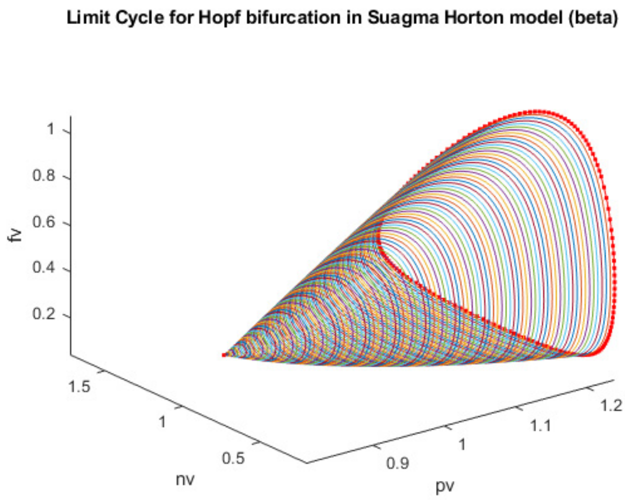

Figure 1c: Limit Cycle in Hopf bifurcation for Sugama Horton model β is bifurcation parameter

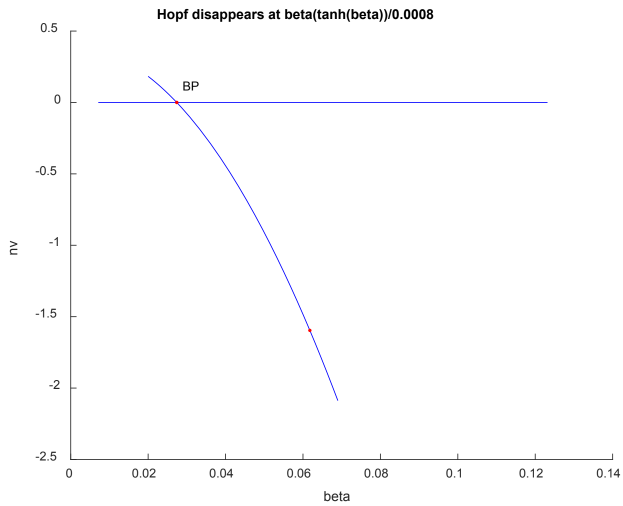

Figure 1d: Sugama Horton model: Hopf bifurcation point disappears when β is modified to $\beta \tanh(\beta)/0.0008$.

When q is the bifurcation parameter, for low values of β (0.15), a Hopf bifurcation point was found at (pv, nv,fv, q) values of (1.238633, 1.106527, 0.789893, 1.713227) **(Figure 1e)**; all other parameter values being the same as the base values. **(Figure 1f)** shows the limit cycle for this Hopf Bifurcation. When q is modified to q(tanh(q)), the Hopf bifurcation point disappears, validating the analysis in Sridhar[28] **(Figure 1g)**. For higher values of β (1.35), a branch point was located at (P, N, F, q) values of (1.514363, 1.402188, 0, 2.654276) **(Figure 1h)**.

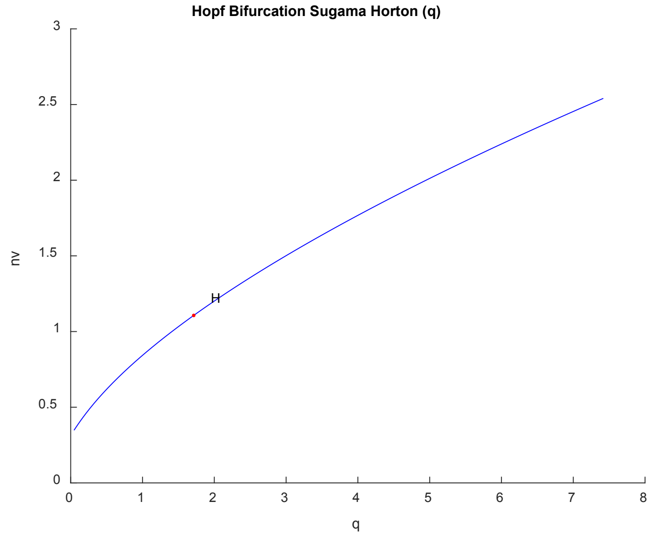

Figure 1e: Sugama Horton model Hopf Bifurcation q is the bifurcation parameter.

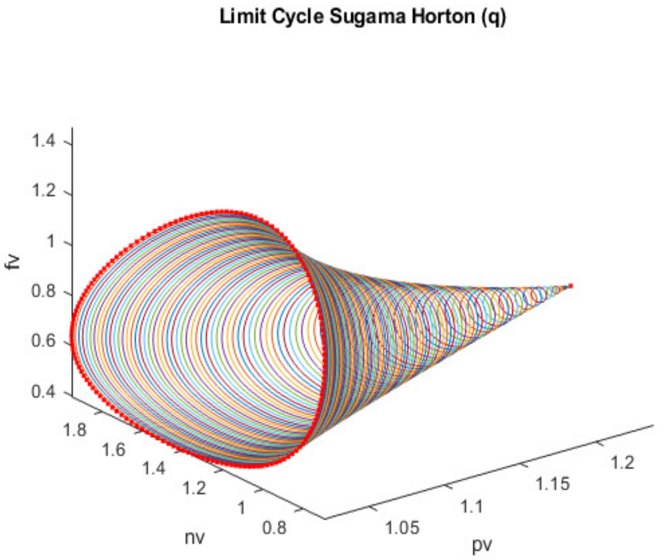

Figure 1f: Limit Cycle for Sugama Horton Model (q is the bifurcation parameter).

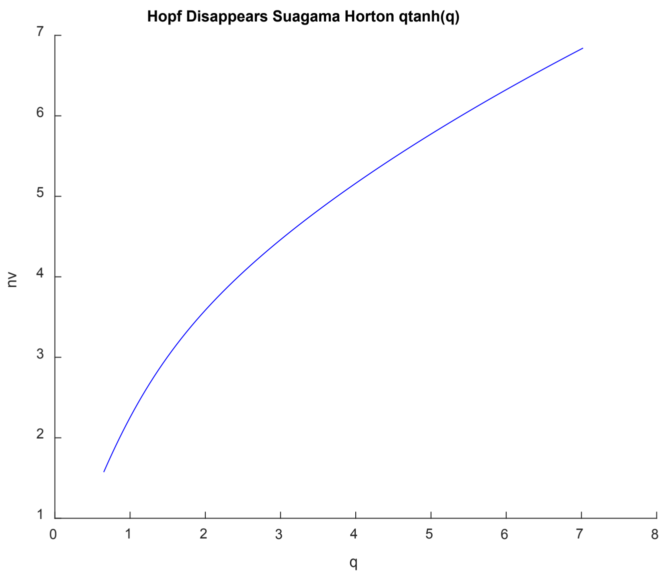

Figure 1g: Hopf bifurcation Disappears Sugama Horton q(tanh(q).

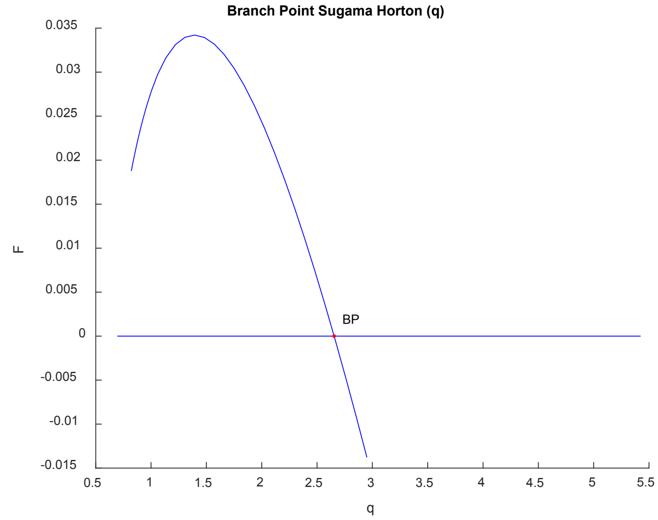

Figure 1h: Sugama Horton model Branch point q is the bifurcation parameter.

13. Bifurcation Analysis (BDS Model)

For the BDS model, when φ is the bifurcation parameter, a Hopf bifurcation point was located at

(pv,nv,fv, φ) values of (2.666679; 2.666679; 0; 0.016866) **(Figure 2a)**. The limit cycle caused by this Hopf bifurcation point, shown in **(Figure 2b)**, is a closed single curve. When φ is modified to $\varphi\tanh(\varphi)$ the Hopf bifurcation point disappears **(Figure 2c)**. The value of q is 7.111175. All other parameter values are base values.

With $\alpha = 2.3; \varphi = 5.e-06$; all other parameter values being base values, and q is the bifurcation parameter, a branch point was located at (pv, nv, fv, q) = (2.756005; 2.756005; 0; 7.595561) **(Figure 2d)**.

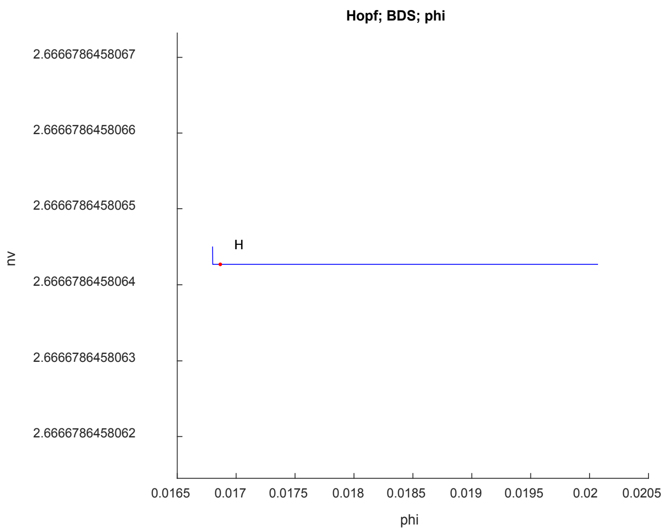

Figure 2a: Hopf Bifurcation; BDS model; φ is bifurcation parameter.

13.1. MNLMPC (Sugama Horton Model)

For the MNLMPC, the procedure described is followed. qv is chosen as the control parameter, and $\sum_{t_i=0}^{t_i=t_f} fv(t_i)$ is maximized and $\sum_{t_{i=0}}^{t_i=t_f} nv(t_i)$ individually, and led to a values of 101.956 and 0. The overall optimal control problem will involve the minimization of $(\sum_{t_{i=0}}^{t_i=t_f} fv(t_i) - 101.956)^2 + (\sum_{t_{i=0}}^{t_i=t_f} nv(t_i) - 0)^2$ Was minimized subject to the model's equations. This led to a value of zero (the Utopia point). The MNLMPC values of the control variable qv is 5.467. The branch points cause the MNLMPC calculations to converge to the Utopia solution, validating the analysis of Sridhar[33]. The control profile qv exhibits spikes, which are remedied using the Savitzky-Golay filter to produce the smooth control profile qvsg. All the MNLMPC profiles are shown in **(Figures 3a and 3b)**.

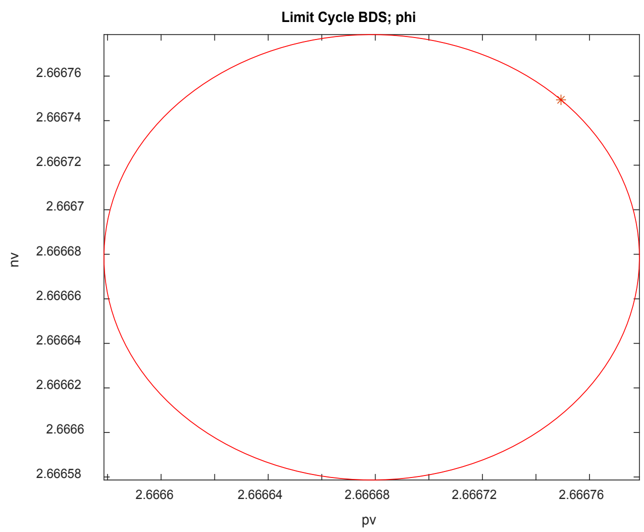

Figure 2b: Limit Cycle caused by Hopf Bifurcation; BDS model; φ is the bifurcation parameter.

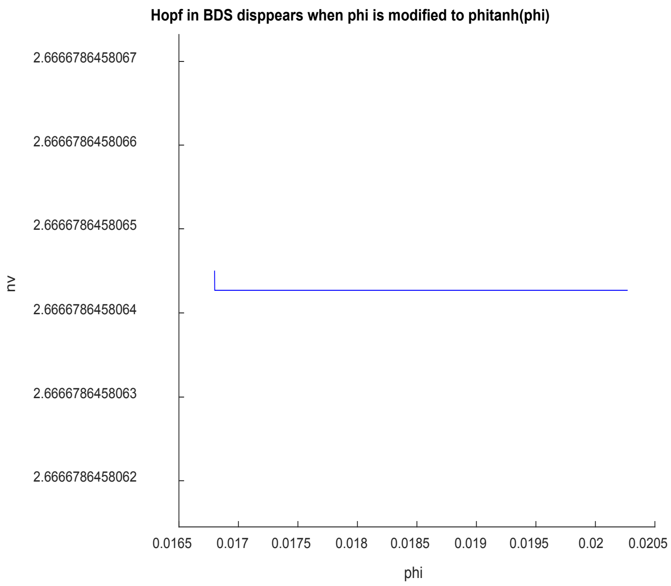

Figure 2c: Hopf Bifurcation; disappears in BDS model; $\varphi \tanh(\varphi)$ is the bifurcation parameter.

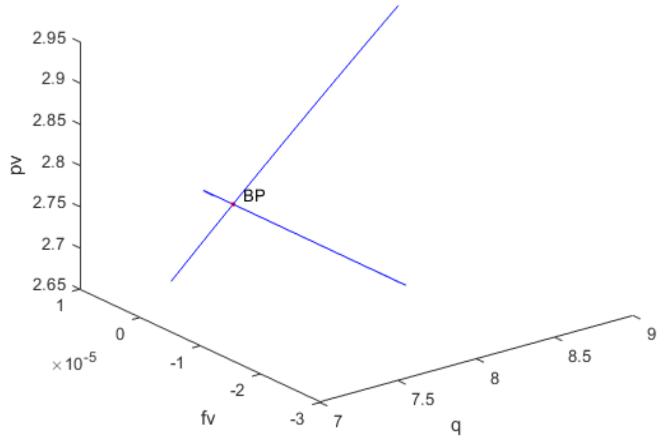

Figure 2d: branch point q is the bifurcation parameter (BDS model).

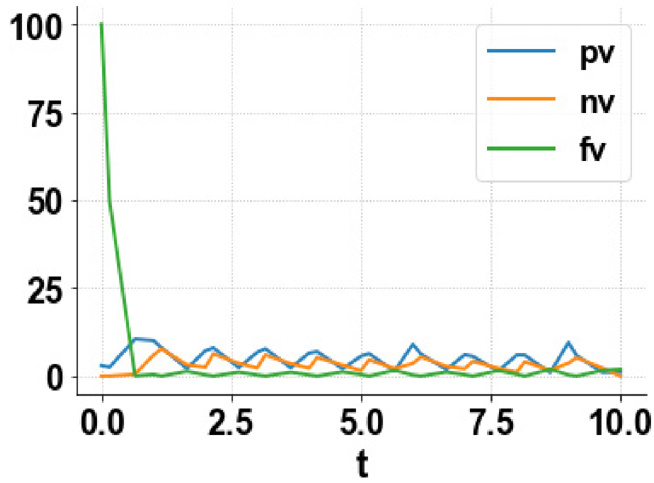

Figure 3a: MNLMPC (Sugama Horton Model; pv; nv fv profiles).

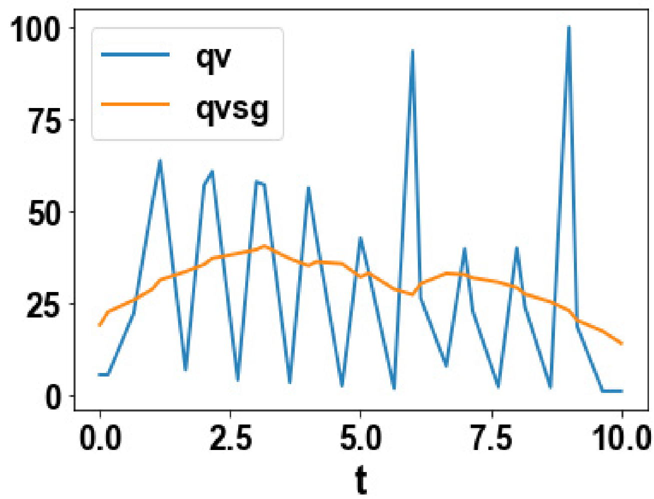

Figure 3b: MNLMPC Sugama Horton Model; qv; qvsg profiles.

13.2. MNLMPC (BDS Model)

For the MNLMPC, the procedure described is followed. qv is chosen as the control parameter, and $\sum_{t_{i=0}}^{t_i=t_f} fv(t_i)$ is maximized and $\sum_{t_{i=0}}^{t_i=t_f} nv(t_i)$ individually, and led to a values of 200 and 0. The overall optimal control problem will involve the minimization of $(\sum_{t_i=0}^{t_i=t_f} fv(t_i)-200)^2+(\sum_{t_i=0}^{t_i=t_f} nv(t_i)-0)^2$ was minimized subject to the model's equations. This led to a value of zero (the Utopia point). The MNLMPC values of the control variable qv is 54.5163. The branch points cause the MNLMPC calculations to converge to the Utopia solution, validating the analysis of Sridhar[34]. The control profile qv exhibits spikes, which are remedied using the Savitzky-Golay filter to produce the smooth control profile qvsg. All the MNLMPC profiles are shown in **(Figures 4a and 4b)**.

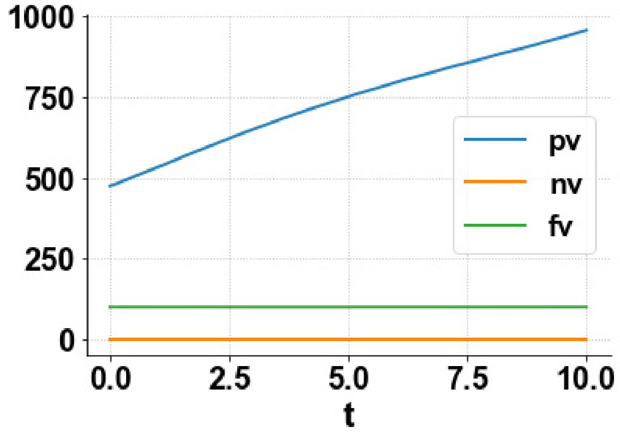

Figure 4a: MNLMPC (BDS Model; pv; nv fv profiles).

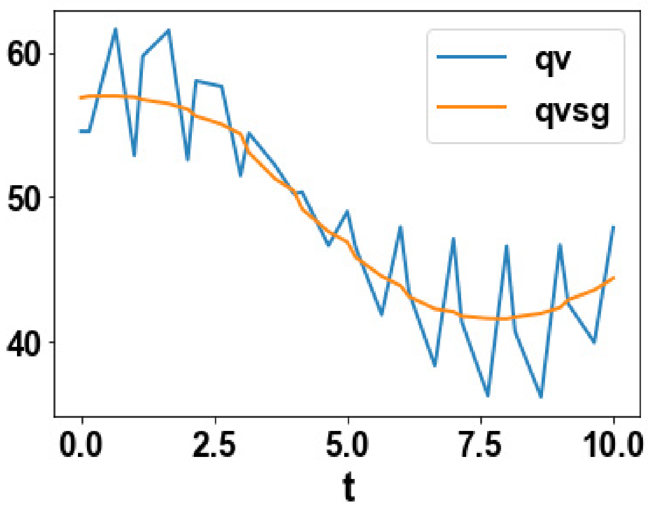

Figure 4b: MNLMPC BDS model; qv; qvsg profiles.

In both models, branch points and Hopf bifurcation points were found. The Branch points are beneficial as they enable the MNLMPC calculations to attain the best possible solution (the Utopia point). The Hopf bifurcation points give rise to unwanted limit cycles and must be eliminated. The activation factor involving the tanh function is shown to effective in eliminating the Hopf bifurcation points in both models.

14. Conclusions

Bifurcation analysis and multiobjective nonlinear control (MNLMPC) studies on the Sugama Horton and the Ball Dewar Sugama plasma turbulence models. The bifurcation analysis revealed Hopf bifurcation points and branch points. Hopf bifurcation points, which cause unwanted limit cycles, are eliminated using an activation function based on the tanh function. The branch points (which cause multiple steady-state solutions from a singular point) are very beneficial because they enable the Multiobjective nonlinear model predictive control calculations to converge to the Utopia point (the best possible solution) in the models. A combination of bifurcation analysis and Multiobjective Nonlinear Model Predictive Control (MNLMPC) for the plasma dynamics modes is the main contribution of this paper.

15. Data Availability Statement

All data used is presented in the paper.

All the data can be reproduced. All the code developed can be reproduced.

16. Conflict of Interest

The author, Dr. Lakshmi N Sridhar, has no conflict of interest.

17. Acknowledgement

Dr. Sridhar thanks Dr. Carlos Ramirez for encouraging him to write single-author papers.

18. Funding

No funding was received for this research.

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