

Analysis and Control of a Dynamic Model That Describes the Avoidance Behavior of Zooplankton on Phytoplankton Infected by Free Viruses

Lakshmi. N. Sridhar*

Chemical Engineering Department, University of Puerto Rico, Mayaguez, PR 00681, USA

Citation: Sridhar LN. Analysis and Control of a Dynamic Model That Describes the Avoidance Behavior of Zooplankton on Phytoplankton Infected by Free Viruses. *J Petro Chem Eng* 2025;3(3):148-154.

Received: 06 September, 2025; **Accepted:** 08 September, 2025; **Published:** 10 September, 2025

***Corresponding author:** Lakshmi N. Sridhar, Chemical Engineering Department, University of Puerto Rico, Mayaguez, PR 00681, USA, Email: lakshmin.sridhar@upr.edu

Copyright: © 2025 Sridhar LN., This is an open-access article published in J Petro Chem Eng (JPCE) and distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.

ABSTRACT

The effect of marine viruses on phytoplankton and zooplankton is highly nonlinear and complicated. The phytoplankton population is infected by external free viruses and zooplankton are affected by the consumption of infected phytoplankton. To minimize the infected phytoplankton and maximize the healthy zooplankton it is important to analyze and control the negative effects of the viruses. In this work, bifurcation analysis and multiobjective nonlinear model predictive control is performed on a dynamic model that describes the avoidance behavior of zooplankton on phytoplankton infected by free viruses. Bifurcation analysis is a powerful mathematical tool used to deal with the nonlinear dynamics of any process. Several factors must be considered and multiple objectives must be met simultaneously. The MATLAB program MATCONT was used to perform the bifurcation analysis. The MNLMPC calculations were performed using the optimization language PYOMO in conjunction with the state-of-the-art global optimization solvers IPOPT and BARON. The bifurcation analysis revealed the existence of branch points, limit points, and Hopf bifurcation points. The Hopf bifurcation point, which causes an unwanted limit cycle, is eliminated using an activation factor involving the tanh function. The MNLMPC converged to the Utopia solution. The branch and limit points (which cause multiple steady-state solutions from a singular point) are very beneficial because they enable the Multiobjective nonlinear model predictive control calculations to converge to the Utopia point (the best possible solution) in the model.

Keywords: Bifurcation; Optimization; Control; Zooplankton; Phytoplankton

Background

Castberg, et al.¹ studied the microbial population dynamics and diversity during a bloom of the marine coccolithophorid *Emiliana huxleyi* (Haptophyta). Jacquet, et al.² performed a flow cytometric analysis of an *Emiliana huxleyi* bloom terminated by viral infection. Chattopadhyay, et al.³ researched the viral infection on the phytoplankton-zooplankton system using a mathematical model. Wilson, et al.⁴ showed that the isolation of viruses was responsible for the demise of an *Emiliana huxleyi* bloom in the English Channel. Chattopadhyay, et al.⁵, conducted preliminary experiments with population dynamics

models of three trophic levels. Singh, et al.⁶ studied the role of virus infection in a simple phytoplankton-zooplankton system. Bairagi, et al.⁷ showed that the virus replication factor may be a controlling agent for obtaining a disease-free system in a multi-species eco-epidemiological system. Evans, et al.⁸ showed that the viral infection of *Emiliana huxleyi* (Prymnesiophyceae) leads to elevated production of reactive oxygen species. Hilker, et al.⁹ demonstrated the existence of oscillations and waves in a virally infected plankton system. Gakkhar, et al.¹⁰ developed a mathematical model for viral infection in toxin-producing phytoplankton and zooplankton systems. Rhodes, et al.¹¹

showed that viral infections could act as a regulator of oceanic phytoplankton populations. Siekmann, et al.¹², I., developed an extension of the Beretta-Kuang model of viral diseases. Evans, et al.¹³ discussed the preferential grazing of *Oxyrrhis marina* on virus-infected *Emiliana huxleyi*. Evans, et al.¹⁴ investigated the changes in *Emiliana huxleyi* fatty acid profiles during infection with the *E. huxleyi* virus and discussed the physiological and ecological implications. Rhodes, et al.¹⁵ studied the influence of viral infection on a plankton ecosystem undergoing nutrient enrichment. Chakraborty, et al.¹⁶ discussed the role of avoidance by zooplankton for survival and dominance of toxic phytoplankton. Samanta, et al.¹⁷ studied the effect of enrichment on plankton dynamics where phytoplankton can be infected from free viruses. Rosenwasser, et al.¹⁸ showed that rewiring host lipid metabolism by large viruses determines the fate of *Emiliana huxleyi*, a bloom-forming alga in the ocean. Gilg, et al.¹⁹ demonstrated that differential gene expression is tied to photochemical efficiency reduction in virally infected *Emiliana huxleyi*. Malitsky, et al.²⁰ showed that the viral infection of the marine alga *Emiliana huxleyi* triggers lipidome remodeling and induces the production of highly saturated triacylglycerol. Costamagna, et al.²¹ developed a model for the operations to render an epidemic-free hog farm infected by the Aujeszky disease. Vermont, et al.²² showed that Virus infection of *Emiliana huxleyi* deters grazing by the copepod *Acartia tonsa*. Biswas, et al.²³ modelled the avoidance behavior of zooplankton on phytoplankton infected by free viruses. This work aims to perform bifurcation analysis and multiobjective nonlinear control (MNLMP) studies on a dynamic model describing the avoidance behavior of zooplankton on phytoplankton infected by free viruses²³. The paper is organized as follows. First, the model equations are presented, followed by a discussion of the numerical techniques involving bifurcation analysis and multiobjective nonlinear model predictive control (MNLMP). The results and discussion are then presented, followed by the conclusions.

Model Equations²³

In this model, *sv*, *iv*, *zv* and *vv* represent the susceptible phytoplankton, infected phytoplankton, zooplankton and free viruses in the environment infecting the phytoplankton. The model equations are

$$\begin{aligned}\frac{d(sv)}{dt} &= a(sv) \left(1 - \left(\frac{(sv+iv)}{k} \right) \right) - \frac{\alpha_1(sv)zv}{(d_1+sv)} - \frac{\beta(sv)vv}{(k_1+vv)} \\ \frac{d(iv)}{dt} &= \left(\frac{\beta(sv)vv}{(k_1+vv)} \right) - \left(\frac{\alpha_2 iv(zv)}{(d_2+iv+(\gamma sv))} \right) - \mu(iv) \\ \frac{d(zv)}{dt} &= \left(\lambda_1 \frac{\alpha_1(sv)zv}{(d_1+sv)} \right) - \left(\lambda_2 \frac{\alpha_2 iv(zv)}{(d_2+iv+(\gamma sv))} \right) - (\nu zv) \\ \frac{d(vv)}{dt} &= b\mu(iv) - \frac{\beta(sv)vv}{(k_1+vv)} - \delta(vv)\end{aligned}$$

The base parameter values are

$$\alpha_1=0.045; d_1=2; \alpha_2=0.045; d_2=2; \gamma=3.8; ; \lambda_1=0.75; \lambda_2=0.61; \nu=0.012; a=0.77; k=108; k_1=3; \beta=0.65; \mu=0.16; b=35; \delta=1.23.$$

Bifurcation analysis

The MATLAB software MATCONT is used to perform the bifurcation calculations. Bifurcation analysis deals with

multiple steady-states and limit cycles. Multiple steady states occur because of the existence of branch and limit points. Hopf bifurcation points cause limit cycles. A commonly used MATLAB program that locates limit points, branch points and Hopf bifurcation points is MATCONT^{24,25}. This program detects Limit points (LP), branch points (BP) and Hopf bifurcation points(H) for an ODE system.

$$\frac{dx}{dt} = f(x, \alpha)$$

$x \in R^n$ Let the bifurcation parameter be α . Since the gradient is orthogonal to the tangent vector,

The tangent plane at any point $W = [w_1, w_2, w_3, w_4, \dots, w_{n+1}]$ must satisfy

$$Aw = 0$$

Where A is

$$A = [\partial f / \partial x \quad | \quad \partial f / \partial \alpha]$$

where $\partial f / \partial x$ is the Jacobian matrix. For both limit and branch points, the Jacobian matrix $J = [\partial f / \partial x]$ must be singular.

For a limit point, there is only one tangent at the point of singularity. At this singular point, there is a single non-zero vector, y , where $Jy=0$. This vector is of dimension n . Since there is only one tangent the vector

$$y = (y_1, y_2, y_3, y_4, \dots, y_n) \quad \text{must align with} \\ \hat{w} = (w_1, w_2, w_3, w_4, \dots, w_n). \text{ Since}$$

$$J\hat{w} = Aw = 0$$

the $n+1$ th component of the tangent vector $w_{n+1} = 0$ at a limit point (LP).

For a branch point, there must exist two tangents at the singularity. Let the two tangents be z and w . This implies that

$$Az = 0$$

$$Aw = 0$$

Consider a vector v that is orthogonal to one of the tangents (say w). v can be expressed as a linear combination of z and w ($v = \alpha z + \beta w$). Since $Az = Aw = 0$; $Av = 0$ and since

w and v are orthogonal, $w^T v = 0$. Hence $Bv = \begin{bmatrix} A \\ w^T \end{bmatrix} v = 0$ which implies that B is singular.

Hence, for a branch point (BP) the matrix $B = \begin{bmatrix} A \\ w^T \end{bmatrix}$ must be singular.

At a Hopf bifurcation point,

$$\det(2f_x(x, \alpha) @ I_n) = 0$$

@ indicates the bialternate product while I_n is the n -square identity matrix. Hopf bifurcations cause limit cycles and should be eliminated because limit cycles make optimization and control tasks very difficult. More details can be found in Kuznetsov^{27,28}.

Hopf bifurcations cause limit cycles. The tanh activation function (where a control value u is replaced by) $(u \tanh u / \varepsilon)$ is used to eliminate spikes in the optimal control profiles²⁹⁻³². Sridhar³³ explained with several examples how the activation factor involving the tanh function also eliminates the Hopf bifurcation points. This was because the tanh function increases the oscillation time period in the limit cycle.

Multiobjective Nonlinear Model Predictive Control (MNLMP)

The rigorous multiobjective nonlinear model predictive control (MNLMP) method developed by Flores Tlacuahuaz, et al.³⁴ was used.

Consider a problem where the variables $\sum_{t_i=0}^{t_i=t_f} q_j(t_i)$ ($j=1, 2..n$) have to be optimized simultaneously for a dynamic problem

$$\frac{dx}{dt} = F(x, u)$$

t_f being the final time value and n the total number of objective variables and u the control parameter. The single objective optimal control problem is solved individually

optimizing each of the variables $\sum_{t_i=0}^{t_i=t_f} q_j(t_i)$. The optimization

of $\sum_{t_i=0}^{t_i=t_f} q_j(t_i)$ will lead to the values q_j^* . Then, the multiobjective optimal control (MOOC) problem that will be solved is

$$\min \left(\sum_{j=1}^n \left(\sum_{t_i=0}^{t_i=t_f} q_j(t_i) - q_j^* \right)^2 \right) \\ \text{subject to } \frac{dx}{dt} = F(x, u);$$

This will provide the values of u at various times. The first obtained control value of u is implemented and the rest are discarded. This procedure is repeated until the implemented and the first obtained control values are the same or if the Utopia

point where $\left(\sum_{t_i=0}^{t_i=t_f} q_j(t_i) = q_j^* \right)$ for all j is obtained.

Pyomo³⁵ is used for these calculations. Here, the differential equations are converted to a Nonlinear Program (NLP) using the orthogonal collocation method. The NLP is solved using IPOPT³⁶ and confirmed as a global solution with BARON³⁷.

The steps of the algorithm are as follows

Optimize $\sum_{t_i=0}^{t_i=t_f} q_j(t_i)$ and obtain q_j^* .

Minimize $\left(\sum_{j=1}^n \left(\sum_{t_i=0}^{t_i=t_f} q_j(t_i) - q_j^* \right)^2 \right)$ and get the control values at various times $s_i=1$ $t_i=0$

Implement the first obtained control values

Repeat steps 1 to 3 until there is an insignificant difference between the implemented and the first obtained value of the control variables or if the Utopia point is achieved. The Utopia

point is when $\sum_{t_i=0}^{t_i=t_f} q_j(t_i) = q_j^*$ for all j .

Sridhar³⁸ demonstrated that when the bifurcation analysis revealed the presence of limit and branch points the MNLMP calculations to converge to the Utopia solution. For this, the singularity condition, caused by the presence of the limit or branch points was imposed on the co-state equation. If the minimization

of q_1 lead to the value q_1^* and the minimization of q_2 lead to the value q_2^* . The MNLMP calculations will minimize the function $(q_1 - q_1^*)^2 + (q_2 - q_2^*)^2$. The multiobjective optimal control problem is

$$\min (q_1 - q_1^*)^2 + (q_2 - q_2^*)^2 \quad \text{subject to } \frac{dx}{dt} = F(x, u)$$

Differentiating the objective function results in

$$\frac{d}{dx_i} ((q_1 - q_1^*)^2 + (q_2 - q_2^*)^2) = 2(q_1 - q_1^*) \frac{d}{dx_i} (q_1 - q_1^*) + 2(q_2 - q_2^*) \frac{d}{dx_i} (q_2 - q_2^*)$$

The Utopia point requires that both $(q_1 - q_1^*)$ and $(q_2 - q_2^*)$ are zero. Hence

$$\frac{d}{dx_i} ((q_1 - q_1^*)^2 + (q_2 - q_2^*)^2) = 0$$

The optimal control co-state equation³⁹ is

$$\frac{d}{dt} (\lambda_i) = - \frac{d}{dx_i} ((q_1 - q_1^*)^2 + (q_2 - q_2^*)^2) - f_x \lambda_i; \quad \lambda_i(t_f) = 0$$

λ_i is the Lagrangian multiplier. t_f is the final time. The first term in this equation is 0 and hence

$$\frac{d}{dt} (\lambda_i) = - f_x \lambda_i; \quad \lambda_i(t_f) = 0$$

At a limit or a branch point, for the set of ODE $\frac{dx}{dt} = f(x, u)$ f_x is singular. Hence there are two different vectors-values for $[\lambda_i]$ where $\frac{d}{dt} (\lambda_i) > 0$ and $\frac{d}{dt} (\lambda_i) < 0$. In between there is a vector $[\lambda_i]$ where $\frac{d}{dt} (\lambda_i) = 0$. This coupled with the boundary condition $\lambda_i(t_f) = 0$ will lead to $[\lambda_i] = 0$. This makes the problem an unconstrained optimization problem and the optimal solution is the Utopia solution.

Results

When a is the bifurcation parameter, a branch point occurred at (sv, iv, zv, vv, a) values of (0; -0.608365; -4.948035; -2.769792; -7.887499) (Figure 1a).

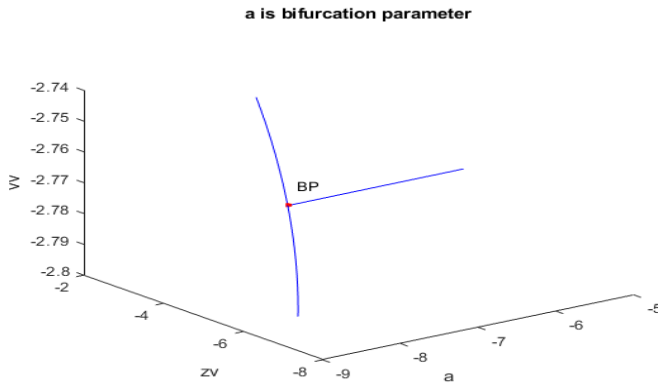


Figure 1a: a is the bifurcation parameter

When $\alpha 1$ is the bifurcation parameter, a branch point occurred at (sv, iv, zv, vv, $\alpha 1$) values of (3.792436; 14.728461; 0; 65.14051; 0.050878) (**Figure 1b**).

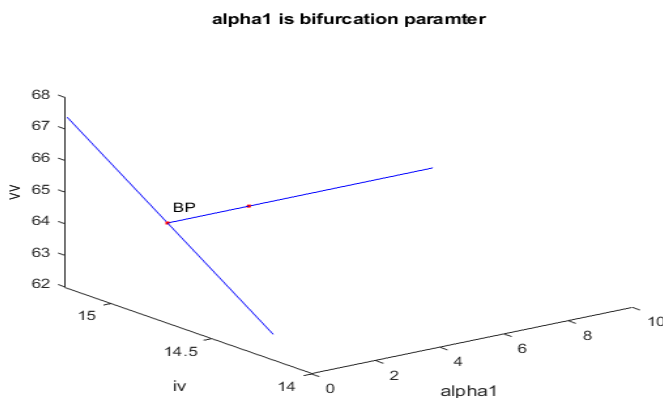


Figure 1b: $\alpha 1$ is the bifurcation parameter.

When $\alpha 2$ is the bifurcation parameter, a Hopf bifurcation point occurs at (sv, iv, zv, vv, $\alpha 2$) values of (19.860109 1.388582 164.282223 2.055185 1.737406). This is indicated in the curve AB in (**Figure 1c**). When $\alpha 2$ is modified to $\alpha 2 \tanh(\alpha 2)/1.5$, the Hopf bifurcation point disappears. This is indicated in the curve CD in Fig. 1c. The limit cycle caused by the Hopf bifurcation is shown in (**Figure 1d**).

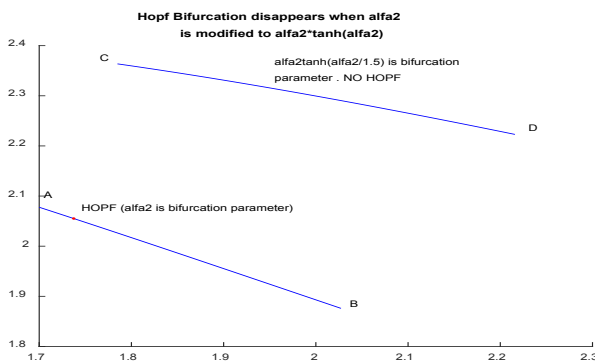


Figure 1c: $\alpha 2$ is the bifurcation parameter, a Hopf bifurcation point is present (curve AB). When $\alpha 2$ is modified to $\alpha 2 \tanh(\alpha 2)/1.5$, the Hopf bifurcation point disappears (curve CD)

When δ was used as the bifurcation parameter several limit points were found. One of them occurred at (sv, iv, zv, vv, δ) values of (0, 0, 0, 310.688616, 0). (**Figure 1e**).

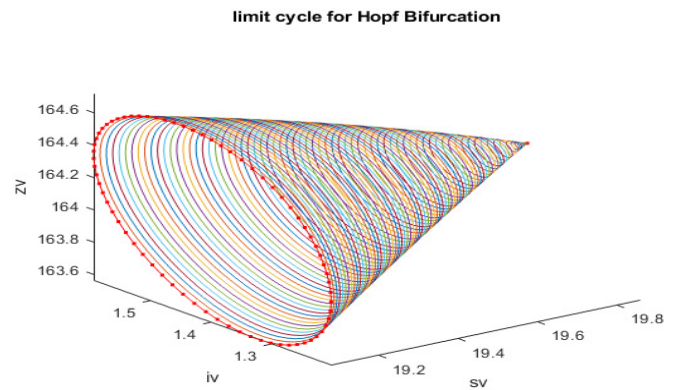


Figure 1d: Limit cycle for Hopf shown in Figure 1d.

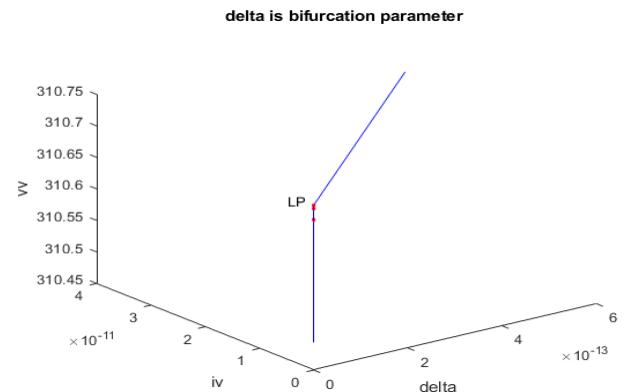


Figure 1e: The bifurcation parameter.

For the MNLMPC calculations, $sv(0) = 200$, $vv(0) = 50$, $zv(0) = 50$, $iv(0) = 50$, $\sum_{t_i=0}^{t_i=t_f} iv(t_i)$ was minimized and $\sum_{t_i=0}^{t_i=t_f} zv(t_i)$ was maximized individually and resulted in values of 50 and 400. The overall optimal control problem will involve the minimization of $(\sum_{t_i=0}^{t_i=t_f} iv(t_i) - 50)^2 + (\sum_{t_i=0}^{t_i=t_f} zv(t_i) - 400)^2$ was subject to the equations governing the model. This led to a value of zero (the Utopia

The MNLMPC values of the control variables, β was 0.000334. The various MNMPC figures are shown in (**Figures 2a-2b**). The control profiles of β exhibited noise and this was remedied using the Savitzky-Golay filter to produce a smoother control profile β_{sg} . Both the noisy and smooth profiles are shown in (**Figure 2c**).

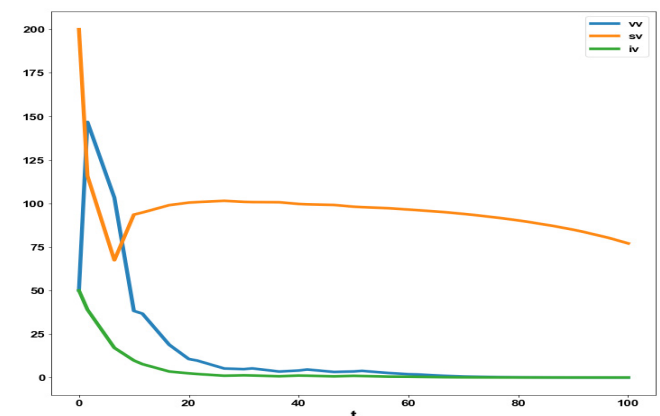


Figure 2a: MNLMPC (vv,sv,iv profiles).

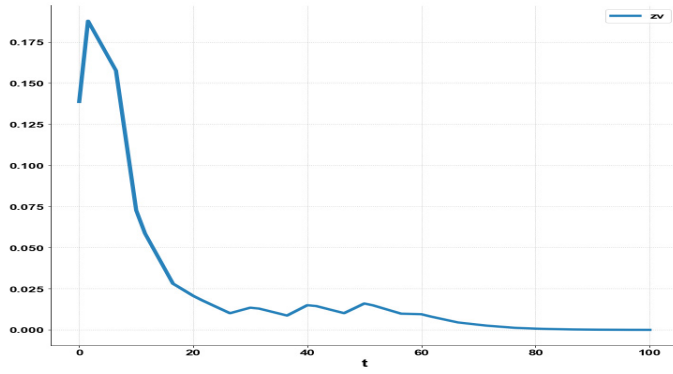


Figure 2b: MNLMPc (zv profile).

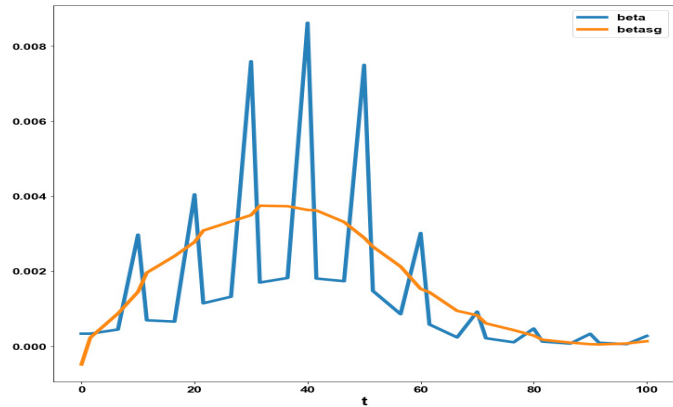


Figure 2c: (β (noise exhibited) β_{sg} (noise eliminated)).

Discussion of Results

Theorem

If at least one of the functions in a dynamic system is separable into two distinct functions, a branch point singularity will occur in the system.

Proof

Consider a system of equations

$$\frac{dx}{dt} = f(x, \alpha)$$

$x \in R^n$. Defining the matrix A as

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} & \dots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial \alpha} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} & \dots & \frac{\partial f_2}{\partial x_n} & \frac{\partial f_2}{\partial \alpha} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \frac{\partial f_n}{\partial x_3} & \frac{\partial f_n}{\partial x_4} & \dots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial \alpha} \end{bmatrix}$$

α is the bifurcation parameter. The matrix A can be written in a compact form as

$$A = \left[\frac{\partial f_p}{\partial x_q} \mid \frac{\partial f_p}{\partial \alpha} \right]$$

The tangent at any point x; ($z = [z_1, z_2, z_3, z_4, \dots, z_{n+1}]$) must satisfy

$$Az = 0$$

The matrix $\left\{ \frac{\partial f_p}{\partial x_q} \right\}$ must be singular at both limit and branch points. The $n+1$ th component of the tangent vector $z_{n+1} = 0$ at a limit point (LP) and for a branch point (BP) the matrix $B = \begin{bmatrix} A \\ z^T \end{bmatrix}$ must be singular.

Any tangent at a point y that is defined by $z = [z_1, z_2, z_3, z_4, \dots, z_{n+1}]$ must satisfy

$$Az = 0$$

For a branch point, there must exist two tangents at the singularity. Let the two tangents be z and w. This implies that

$$Az = 0$$

$$Aw = 0$$

Consider a vector v orthogonal to one of the tangents (z). v can be expressed as a linear combination of z and w ($v = \alpha z + \beta w$). Since $Az = Aw = 0$; $Av = 0$ and since z and v are orthogonal, $z^T v = 0$. Hence $Bv = \begin{bmatrix} A \\ z^T \end{bmatrix} v = 0$

which implies that B is singular where $B = \begin{bmatrix} A \\ z^T \end{bmatrix}$

Let any of the functions f_i are separable into 2 functions ϕ_1, ϕ_2 as

$$f_i = \phi_1 \phi_2$$

At steady-state $f_i(x, \alpha) = 0$. Hence either $\phi_1 = 0$ or $\phi_2 = 0$ or both ϕ_1 and ϕ_2 must be 0 implying that the two branches $\phi_1 = 0$ and $\phi_2 = 0$ will intersect at a point ϕ_1 and ϕ_2 are 0.

Here, B will be singular as one row in this matrix would be

$$\left[\frac{\partial f_i}{\partial x_k} \mid \frac{\partial f_i}{\partial \alpha} \right]$$

However,

$$\left[\frac{\partial f_i}{\partial x_k} = \phi_1 (=0) \frac{\partial \phi_2}{\partial x_k} + \phi_2 (=0) \frac{\partial \phi_1}{\partial x_k} = 0 (\forall k = 1, \dots, n) \right]$$

$$\frac{\partial f_i}{\partial \alpha} = \phi_1 (=0) \frac{\partial \phi_2}{\partial \alpha} + \phi_2 (=0) \frac{\partial \phi_1}{\partial \alpha} = 0$$

This implies that every element in the row $\left[\frac{\partial f_i}{\partial x_k} \mid \frac{\partial f_i}{\partial \alpha} \right]$ would be 0 and hence the matrix B would be singular, implying a branch point.

When α is the bifurcation parameter, a branch point occurred at (sv, iv, zv, vv, a) values of (0; -0.608365; -4.948035; -2.769792; -7.887499).

Here, the two distinct functions can be obtained from the first ODE in the model

$$\frac{d(sv)}{dt} = a(sv) \left(1 - \left(\frac{(sv+iv)}{k} \right) \right) - \frac{\alpha 1(sv)zv}{(d1+sv)} - \frac{\beta(sv)vv}{(k1+vv)}$$

The two distinct equations are

$$sv = 0$$

$$a \left(1 - \left(\frac{(iv)}{k} \right) \right) - \frac{\alpha 1zv}{(d1)} - \frac{\beta vv}{(k1+vv)} = 0$$

With $sv=0$; $a = -7.887499$; $iv = -0.608365$; $zv = -4.948035$; $vv = -2.769792$; $k = 108$; $\alpha 1 = 0.045$; $d1 = 2$; $\beta = 0.65$; $k1 = 3$; both the distinct equations are satisfied, validating the theorem.

When $\alpha 1$ is the bifurcation parameter, a branch point occurred at $(sv, iv, zv, vv, \alpha 1)$ values of $(3.792436; 14.728461; 0; 65.14051; 0.050878)$

Here, the two distinct functions can be obtained from the third ODE in the model

$$\frac{d(zv)}{dt} = \left(\lambda 1 \frac{\alpha 1(sv)zv}{(d1+sv)} \right) - \left(\lambda 2 \frac{\alpha 2iv(zv)}{(d2+iv+(\gamma sv))} \right) - (\nu zv)$$

The two distinct equations are

$$zv = 0$$

$$\left(\lambda 1 \frac{\alpha 1(sv)}{(d1+sv)} \right) - \left(\lambda 2 \frac{\alpha 2(iv)}{(d2+iv+(\gamma sv))} \right) - \nu = 0$$

With

$$\alpha 1 = 0.050878; d1 = 2; \alpha 2 = 0.045; d2 = 2; \gamma = 3.8; \lambda 1 = 0.75; \lambda 2 = 0.61; \nu = 0.012; sv = 3.792436; iv = 14.728461; zv = 0; vv = 65.14051$$

both the distinct equations are satisfied, validating the theorem.

When $\alpha 2$ is the bifurcation parameter, a Hopf bifurcation point occurs at $(sv, iv, zv, vv, \alpha 2)$ values of $(19.860109, 1.388582, 164.282223, 2.055185, 1.737406)$. When $\alpha 2$ is modified to $\alpha 2 \tanh(\alpha 2)/1.5$, the Hopf bifurcation point disappears. This validates the analysis of Sridhar³³.

It is seen that the presence of the limit and branch points are beneficial because it allows the MNLMPC calculations to attain the Utopia solution, validating the analysis of Sridhar³⁸.

Conclusions

Bifurcation analysis and multiobjective nonlinear model predictive control (MNLMPC) studies were conducted on a dynamic model that describes the avoidance behavior of zooplankton on phytoplankton infected by free viruses. The bifurcation analysis revealed the existence of branch points, limit points and Hopf bifurcation points in the model. The branch and limit points (which cause multiple steady-state solutions from a singular point) are very beneficial because they enable the Multiobjective nonlinear model predictive control calculations to converge to the Utopia point (the best possible solution). It is proven (with computational validation) that the branch points were caused by the existence of two distinct separable functions in one of the equations in each dynamic model. A theorem was

developed to demonstrate this fact for any dynamic model. The Hopf bifurcation point that causes an unwanted limit cycle is eliminated using an activation factor involving the tanh function. A combination of bifurcation analysis and Multiobjective Nonlinear Model Predictive Control (MNLMPC) for a dynamic model that describes the avoidance behaviour of zooplankton on phytoplankton infected by free viruses is the main contribution of this paper.

Data availability statement

All data used is presented in the paper.

Conflict of interest

The author, Dr. Lakshmi N Sridhar, has no conflict of interest.

Acknowledgement

Dr. Sridhar thanks Dr. Carlos Ramirez and Dr. Suleiman for encouraging him to write single-author papers

References

1. Castberg T, et al. Microbial population dynamics and diversity during a bloom of the marine coccolithophorid *Emiliania huxleyi* (Haptophyta). *Mar Ecol Prog Ser* 2001;221:39-46.
2. Jacquet S, Heldal M, Rodriguez DI, et al. Flow cytometric analysis of an *Emiliania huxleyi* bloom terminated by viral infection. *Aquat. Microb. Ecol.* 2002;27:111-124.
3. Chattopadhyay J, Pal S. Viral infection on phytoplankton-zooplankton system: a mathematical model. *Ecol Model* 2002;151:15-28.
4. Wilson WH, Tarran GA, Schroeder D, et al. Isolation of viruses responsible for the demise of an *Emiliania huxleyi* bloom in the English Channel. *J Mar Biol Assoc* 2002;82(3):369-377.
5. Chattopadhyay J, Sarkar RR. Chaos to order: preliminary experiments with a population dynamics models of three trophic levels. *Ecol Model* 2003;163:45-50.
6. Singh BK, Chattopadhyay J, Sinha S. The role of virus infection in a simple phytoplankton zooplankton system. *J Theor Biol* 2004;231:153-166.
7. Bairagi N. Virus replication factor may be a controlling agent for obtaining disease-free system in a multi-species eco-epidemiological system. *J Biol Syst* 2005;13(3):245-259.
8. Evans C, Malin G, Mills GP, et al. Viral infection of *Emiliania huxleyi* (Prymnesiophyceae) leads to elevated production of reactive oxygen species. *J Phycol* 2006;42:1040-1047.
9. Hilker FM, Malchow H, Langlais M, et al. Oscillations and waves in a virally infected plankton system Part II: Transition from lysogeny to lysis. *Ecol Compl* 2006;3:200-208.
10. Gakkhar S, Negi K. A mathematical model for viral infection in toxin producing phytoplankton and zooplankton system. *Appl Math Comp* 2006;179:301-313.
11. Rhodes C, Truscott J, Martin A. Viral infection as a regulator of oceanic phytoplankton populations. *J Mar Syst* 2008;74:216-226.
12. Siekmann I, Malchow H, Venturino E. An extension of the Beretta-Kuang model of viral diseases. *Math Biosci Eng* 2008;5:549-565.
13. Evans C, Wilson WH. Preferential grazing of *Oxyrrhis marina* on virus infected *Emiliania huxleyi*. *Limnol Oceanogr* 2008;53:2035-2040.
14. Evans C, Pond DW, Wilson WH. Changes in *Emiliania huxleyi* fatty acid profiles during infection with *E. huxleyi* virus 86:

- physiological and ecological implications. *Aquat Microb Ecol* 2009;55:219-228.
15. Rhodes CJ, Martin AP. The influence of viral infection on a plankton ecosystem undergoing nutrient enrichment. *J Theor Biol* 2010;265(3):225-237.
 16. Chakraborty S, Bhattacharya S, Feudel U, et al. The role of avoidance by zooplankton for survival and dominance of toxic phytoplankton. *Ecol Compl* 2012;11:144-153.
 17. Samanta S. Effect of enrichment on plankton dynamics where phytoplankton can be infected from free viruses. *Nonlinear Stud* 2013;20(2):223-236.
 18. Rosenwasser S, Mausz MA, Schatz D, et al. Rewiring host lipid metabolism by large viruses determines the fate of *Emiliana huxleyi*, a bloom-forming alga in the ocean. *Plant Cell* 2014.
 19. Gilg IC, Archer SD, Floge SA, et al. Differential gene expression is tied to photochemical efficiency reduction in virally infected *Emiliana huxleyi*. *Mar Ecol Prog Ser* 2016;555:13-27.
 20. Malitsky S, Ziv C, Rosenwasser S, et al. Viral infection of the marine alga *Emiliana huxleyi* triggers lipidome remodeling and induces the production of highly saturated triacylglycerol. *New Phytol* 2016;210(1):88-96.
 21. Costamagna A, Drigo M, Martini M, et al. A model for the operations to render epidemic-free a hog farm infected by the Aujeszky disease. *Appl Math Nonlinear Sci* 2016;1(1):207-228.
 22. Vermont A, Martinez JM, Waller JD, et al. Virus infection of *Emiliana huxleyi* deters grazing by the copepod *Acartia tonsa*. *J Plankton Res* 2016;38(5):1194-1205.
 23. Biswas S, Tiwari PK, Bona F, Pal S, Venturino E. Modeling the avoidance behavior of zooplankton on phytoplankton infected by free viruses. *J Biol Phys* 2020;46(1):1-31.
 24. Dhooge A, Govaerts W, Kuznetsov AY. MATCONT: A Matlab package for numerical bifurcation analysis of ODEs. *ACM transactions on Mathematical software* 2003;29(2):141-164.
 25. Dhooge A, Govaerts W, Kuznetsov YA, Mestrom W, Riet AM. CL_MATCONT; A continuation toolbox in Matlab 2004.
 26. Kuznetsov YA. Elements of applied bifurcation theory. Springer, NY 1998.
 27. Kuznetsov YA. Five lectures on numerical bifurcation analysis. Utrecht University, NL 2009.
 28. Govaerts WJF. Numerical Methods for Bifurcations of Dynamical Equilibria. SIAM 2000.
 29. Dubey SR, Singh SK, Chaudhuri BB. Activation functions in deep learning: A comprehensive survey and benchmark. *Neurocomputing* 2022;503:92-108.
 30. Kamalov F, Nazir A, Safaraliev M, Cherukuri AK, Zgheib R. Comparative analysis of activation functions in neural networks. 28th IEEE Int Conf on Electronics, Circuits and Systems (ICECS) 2021:1-6.
 31. Szandala T. Review and Comparison of Commonly Used Activation Functions for Deep Neural Networks. ArXiv 2020.
 32. Sridhar LN. Bifurcation Analysis and Optimal Control of the Tumor Macrophage Interactions. *Biomed J Sci Tech Res* 2023;53(5).
 33. Sridhar LN. Elimination of oscillation causing Hopf bifurcations in engineering problems. *J App Math* 2024b;2(4):1826.
 34. Flores-Tlacuahuac, A. Pilar Morales, Toledo MR. Multiobjective Nonlinear model predictive control of a class of chemical reactors. *I & EC research* 2012:5891-5899.
 35. William HE, Laird CD, Watson JP, et al. Sirola. Pyomo - Optimization Modeling in Python Second Edition 67.
 36. Wächter A, Biegler L. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Math Program* 2006;106:25-57.
 37. Tawarmalani M, Sahinidis NV. A polyhedral branch-and-cut approach to global optimization. *Mathematical Programming* 2005;103(2):225-249.
 38. Sridhar LN. Coupling Bifurcation Analysis and Multiobjective Nonlinear Model Predictive Control. *Austin Chem Eng* 2024;10(3):1107.
 39. Upreti, Simant Ranjan. Optimal control for chemical engineers. Taylor and Francis 2013.